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On the Landau–Lifschitz Degrees of Freedom in 2-D Turbulence

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We show that if the Kraichnan theory of fully developed turbulence holds, then the Landau–Lifschitz degrees of freedom is bounded (up to a logarithmic term) by $G^{1/2}$, where G is the Grashof number.

KEY WORDS: Navier–Stokes; turbulence.

The incompressible Navier–Stokes equations (NSE) with periodic boundary conditions on $[0, L]^2$ can be written as

$$\frac{du}{dt} + \nu A u + B(u, u) = f \tag{1.1}$$

where $A = -\Delta$, $B(u, v) = \mathscr{P}((u \cdot \nabla)\, v)$ with $\mathscr{P}$ the Helmholtz–Leray projection onto divergence free functions, and f is a body force. We assume that $f = P_{\bar{\kappa}} f$, where

$$P_\kappa u = \sum_{\kappa_0 |k| \leqslant \kappa} \hat{u}_k e^{i\kappa_0 k \cdot x}, \qquad \text{for} \quad u(x) = \sum_{k \in \mathbb{Z}^2} \hat{u}_k e^{i\kappa_0 k \cdot x},$$

$$\text{with} \quad \kappa_0 = 2\pi/L,$$

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and that $\bar{\kappa}/\kappa_0 \leqslant C_0$. Critical wave numbers κ_η, κ_σ, are defined through the generalized time averages (see ref. 1)

$$\eta = \frac{\nu}{L^2}\langle |Au|^2\rangle, \quad \epsilon = \frac{\nu}{L^2}\langle |A^{1/2}u|^2\rangle \quad \text{as} \quad \kappa_\eta = \left(\frac{\eta}{\nu^3}\right)^{1/6} \quad \text{and} \quad \kappa_\sigma = \left(\frac{\eta}{\epsilon}\right)^{1/2}, \tag{1.2}$$

where $|\cdot|$ is the L^2-norm.

It is shown in ref. 1 that if the Kraichnan theory of fully developed turbulence[2] holds for the NSE, then

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^2 \leqslant \left(\frac{1}{2\pi}\right)^{2/3} \left[\left(\frac{\kappa_\sigma}{\bar{\kappa}}\right)^2 - 1\right]^{-1/3} G^{2/3}, \tag{1.3}$$

where $G = |f|/(\nu\kappa_0)^2$ is the Grashof number. The ratio $(\kappa_\eta/\kappa_0)^2$ is the Landau–Lifschitz asymptotic degrees of freedom, which is shown in ref. 3 to be an upper bound on $\dim_F(\mathscr{A})$, the fractal dimension of the global attractor[4] (up to a logarithmic term in κ_η/κ_0)). We also show in ref. 1 that if the Kraichnan theory holds, then

$$\kappa_\sigma \sim \kappa_\eta (\ln \kappa_\eta/\underline{\kappa}_{\mathrm{i}})^{-1/2}, \tag{1.4}$$

where $\underline{\kappa}_{\mathrm{i}}$ is the lower endpoint of the inertial range. Using (1.4) in (1.3) leads in ref. 1 to the somewhat surprising estimate $(\kappa_\eta/\kappa_0)^2 \lesssim G^{4/7}$ (up to a logarithmic term). This undercuts the previous best estimate $(\kappa_\eta/\kappa_0)^2 \lesssim G^{2/3}$ (up to a logarithmic term), made in ref. 3 without assuming turbulence.

The power 4/7 does not, however, fully exploit the relations (1.4) and (1.3). In fact, we show in the next few lines that $(\kappa_\eta/\kappa_0)^2 \lesssim G^{1/2}$ (up to a logarithmic term).

Use (1.4) in (1.3) to obtain

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^6 \left[\left(\frac{\kappa_\eta}{\kappa_0}\right)^2 \left(\frac{\kappa_0}{\bar{\kappa}}\right)^2 \left(\ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}\right)^{-1} - 1\right] \lesssim G^2.$$

Apply the estimate $\kappa_\eta/\kappa_0 \leqslant G^{1/3}$ from ref. 5 to reach

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^8 \left(\frac{\kappa_0}{\bar{\kappa}}\right)^2 \left(\ln \frac{\kappa_\eta}{\underline{\kappa}_{\mathrm{i}}}\right)^{-1} \lesssim G^2 + \left(\frac{\kappa_\eta}{\kappa_0}\right)^6 \leqslant 2G^2,$$

from which immediately follows

$$\left(\frac{\kappa_\eta}{\kappa_0}\right)^2 \left(\ln \frac{\kappa_\eta}{\underline{\kappa}_{\rm i}}\right)^{-1/4} \lesssim \left(\frac{\bar{\kappa}}{\kappa_0}\right)^{1/2} G^{1/2}. \tag{1.5}$$

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